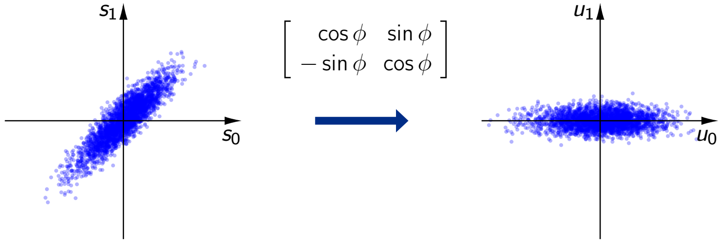
Transform Coding II



What Is The Optimal Orthogonal Transform?

Optimal Orthogonal Transform for General Stationary Signals

- Difficult interdependencies between transform and scalar quantization
- Optimal transform difficult to determine (does depend on bit rate)
- Iterative algorithms for designing both transform and scalar quantizers

Nearly Optimal Transform

- Remember: Important to uitilize dependencies between samples
- Linear transform: Can only remove linear dependendencies (correlation)

➡ Design criterion

$$\boldsymbol{C}_{UU} = \begin{bmatrix} \sigma_0^2 & 0 & \cdots & 0 \\ 0 & \sigma_1^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{N-1}^2 \end{bmatrix}$$

➡ **Karhunen Loève Transform (KLT)**

- Also known as Hotelling transform, eigenvector transform
- Closely related to principle component analysis (PCA)

Karhunen Loève Transform (KLT)

- Remember: Relationship between covariance matrices

$$\begin{aligned} \boldsymbol{C}_{UU} &= \mathrm{E}\Big\{ (\boldsymbol{U} - \mathrm{E}\{\boldsymbol{U}\})\,(\boldsymbol{U} - \mathrm{E}\{\boldsymbol{U}\})^{\mathrm{T}} \Big\} \\ &= \mathrm{E}\Big\{ (\boldsymbol{A}\boldsymbol{S} - \mathrm{E}\{\boldsymbol{A}\boldsymbol{S}\})\,(\boldsymbol{A}\boldsymbol{S} - \mathrm{E}\{\boldsymbol{A}\boldsymbol{S}\})^{\mathrm{T}} \Big\} \\ &= \boldsymbol{A} \cdot \mathrm{E}\Big\{ (\boldsymbol{S} - \mathrm{E}\{\boldsymbol{S}\})\,(\boldsymbol{S} - \mathrm{E}\{\boldsymbol{S}\})^{\mathrm{T}} \Big\} \cdot \boldsymbol{A}^{\mathrm{T}} \\ &= \boldsymbol{A} \cdot \boldsymbol{C}_{SS} \cdot \boldsymbol{A}^{\mathrm{T}} \end{aligned} \tag{1}$$

Karhunen Loève Transform (KLT)

- Orthogonal transform ($\boldsymbol{A}^{-1} = \boldsymbol{A}^{\mathrm{T}}$)
- Produces completely decorrelated transform coefficients

➡ Transform matrix $\boldsymbol{A}$ in chosen in a way that

$$\boldsymbol{C}_{UU} = \boldsymbol{A} \cdot \boldsymbol{C}_{SS} \cdot \boldsymbol{A}^{\mathrm{T}} = \boldsymbol{A} \cdot \boldsymbol{C}_{SS} \cdot \boldsymbol{A}^{-1}$$

becomes a diagonal matrix

Karhunen Loève Transform (KLT)

- By multiplying equation

$$\boldsymbol{C}_{UU} = \boldsymbol{A} \cdot \boldsymbol{C}_{SS} \cdot \boldsymbol{A}^{\mathrm{T}}$$

with $\boldsymbol{A}^{\mathrm{T}}$ from the front (note: $\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}} = \boldsymbol{I}$), we get

$$\boldsymbol{C}_{SS} \cdot \boldsymbol{A}^{\mathrm{T}} = \boldsymbol{A}^{\mathrm{T}} \cdot \boldsymbol{C}_{UU} \tag{2}$$

➡ Required property in matrix form

$$\boldsymbol{C}_{SS} \begin{bmatrix} | & | & & | \\ \boldsymbol{b}_0 & \boldsymbol{b}_1 & \cdots & \boldsymbol{b}_{N-1} \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \boldsymbol{b}_0 & \boldsymbol{b}_1 & \cdots & \boldsymbol{b}_{N-1} \\ | & | & & | \end{bmatrix} \begin{bmatrix} \sigma_0^2 & 0 & \cdots & 0 \\ 0 & \sigma_1^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{N-1}^2 \end{bmatrix} \tag{3}$$

➡ Consider equation of each basis vector $\boldsymbol{b}_k$

Karhunen Loève Transform (KLT)

- For all basis vectors $\boldsymbol{b}_k$, we have an equation of the form

$$\boldsymbol{C}_{SS} \cdot \boldsymbol{b}_k = \sigma_k^2 \cdot \boldsymbol{b}_k \qquad \Longleftrightarrow \qquad (\boldsymbol{C}_{SS} - \xi\, \boldsymbol{I}) \cdot \boldsymbol{v} = \boldsymbol{0} \tag{4}$$

➡ Eigenvector equation with eigenvector $\boldsymbol{v} = \boldsymbol{b}_k$ and eigenvalue $\xi = \sigma_k^2$!

Transform Matrix of KLT

➡ Basis vectors $\boldsymbol{b}_k = \boldsymbol{v}_k / \|\boldsymbol{v}_k\|$ are the **unit-norm eigenvectors** of $\boldsymbol{C}_{SS}$

➡ Transform coefficient variances $\sigma_k^2 = \xi_k$ are the associated eigenvalues of $\boldsymbol{C}_{SS}$

$$\boldsymbol{A} = \begin{bmatrix} - & \boldsymbol{b}_0 & - \\ - & \boldsymbol{b}_1 & - \\ & \vdots & \\ - & \boldsymbol{b}_{N-1} & - \end{bmatrix} \qquad \boldsymbol{C}_{UU} = \begin{bmatrix} \sigma_0^2 & 0 & \cdots & 0 \\ 0 & \sigma_1^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{N-1}^2 \end{bmatrix} \tag{5}$$

Existence and Determination of KLT

Existence and Uniqueness

- KLT exists for all random sources, since symmetric matrices (such as $\boldsymbol{C}_{SS}$) are always orthogonally diagonizable
- There are multiple KLT transform matrices
  - ➡ Rows can be permuted or multiplied by -1
  - ➡ Additional degrees of freedom if eigenvalues are not distinct
- Typically: Basis vectors are sorted according to decreasing eigenvalues

Determination of Transform Matrix

- Solve characteristic polynomial

$$\det(\boldsymbol{C}_{SS} - \xi \cdot \boldsymbol{I}) = 0 \tag{6}$$

- Given ξ_k, solve equation (non-trivial solution with $\|\boldsymbol{v}_k\| = 1$)

$$(\boldsymbol{C}_{SS} - \xi_k \cdot \boldsymbol{I}) \cdot \boldsymbol{v}_k = \boldsymbol{0} \tag{7}$$

Example: KLT for $N = 2$

- Given: Autocovariance matrix (of order 2) for source samples

$$\boldsymbol{C}_{SS} = \begin{bmatrix} \sigma_S^2 & \varrho\,\sigma_S^2 \\ \varrho\,\sigma_S^2 & \sigma_S^2 \end{bmatrix}$$

➡ Eigenvector equation

$$(\boldsymbol{C}_{SS} - \xi\,\boldsymbol{I})\,\boldsymbol{v} = \begin{bmatrix} \sigma_S^2 - \xi & \varrho\,\sigma_S^2 \\ \varrho\,\sigma_S^2 & \sigma_S^2 - \xi \end{bmatrix} \boldsymbol{v} = \boldsymbol{0}$$

➡ Characteristic polynomial

$$\begin{aligned} \det(\boldsymbol{C}_{SS} - \xi\,\boldsymbol{I}) &= \left(\sigma_S^2 - \xi\right)^2 - (\varrho\,\sigma_S^2)^2 \\ &= \xi^2 - 2\xi \cdot \sigma_S^2 + \sigma_S^4\,(1 - \varrho^2) = 0 \end{aligned}$$

➡ Eigenvalues

$$\begin{aligned} \xi_{0/1} &= \sigma_S^2 \pm \sqrt{\sigma_S^4 - \sigma_S^4\,(1 - \varrho^2)} = \sigma_S^2 \pm \sqrt{\sigma_S^4\,\varrho^2} \\ \xi_{0/1} &= \sigma_S^2\,(1 \pm \varrho) \end{aligned}$$

Example: KLT for $N = 2$

- Eigenvector equation for first eigenvalue $\xi_0 = \sigma_S^2\,(1+\varrho)$

$$(\boldsymbol{C}_{SS} - \xi_0\,\boldsymbol{I})\;\boldsymbol{v}_0 \;=\; \begin{bmatrix} \sigma_S^2 - \sigma_S^2(1+\varrho) & \varrho\,\sigma_S^2 \\ \varrho\,\sigma_S^2 & \sigma_S^2 - \sigma_S^2(1+\varrho) \end{bmatrix} \boldsymbol{v}_0$$

$$= \sigma_S^2 \begin{bmatrix} -\varrho & \varrho \\ \varrho & -\varrho \end{bmatrix} \boldsymbol{v}_0 = \boldsymbol{0}$$

- Equation for vector components $\boldsymbol{v}_0 = (u_0, v_0)$

$$-\varrho \cdot u_0 + \varrho \cdot v_0 = 0 \qquad \Longrightarrow \qquad v_0 = u_0$$

- Eigenvector

$$\boldsymbol{v}_0 = \mu \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{with} \qquad \mu \neq 0$$

- Basis vector $\boldsymbol{b}_0$ is given by a unit-norm eigenvector

$$\boldsymbol{b}_0 = \frac{\boldsymbol{v}_0}{\|\boldsymbol{v}_0\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Example: KLT for $N = 2$

- Eigenvector equation for second eigenvalue $\xi_1 = \sigma_S^2\,(1-\varrho)$

$$(\boldsymbol{C}_{SS} - \xi_1\,\boldsymbol{I})\;\boldsymbol{v}_0 = \begin{bmatrix} \sigma_S^2 - \sigma_S^2(1-\varrho) & \varrho\,\sigma_S^2 \\ \varrho\,\sigma_S^2 & \sigma_S^2 - \sigma_S^2(1-\varrho) \end{bmatrix} \boldsymbol{v}_1$$

$$= \sigma_S^2 \begin{bmatrix} \varrho & \varrho \\ \varrho & \varrho \end{bmatrix} \boldsymbol{v}_1 = \boldsymbol{0}$$

- Equation for vector components $\boldsymbol{v}_1 = (u_1, v_1)$

$$\varrho \cdot u_1 + \varrho \cdot v_1 = 0 \qquad \Longrightarrow \qquad v_1 = -u_1$$

- Eigenvector

$$\boldsymbol{v}_1 = \mu \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \text{with} \qquad \mu \neq 0$$

- Basis vector $\boldsymbol{b}_1$ is given by a unit-norm eigenvector

$$\boldsymbol{b}_1 = \frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Example: KLT for $N = 2$

Summary of Example

- Eigenvalues

$$\xi_0 = \sigma_S^2\,(1+\varrho) \qquad \text{and} \qquad \xi_1 = \sigma_S^2\,(1-\varrho)$$

- Eigenvectors

$$\boldsymbol{v}_0 = \mu \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{v}_1 = \mu \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \text{with} \qquad \mu \neq 0$$

➡ Basis vectors and transform matrix

$$\boldsymbol{A} = \begin{bmatrix} - \boldsymbol{b}_0 - \\ - \boldsymbol{b}_1 - \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

➡ Covariance matrix of transform coefficients

$$\boldsymbol{C}_{SS} = \sigma_S^2 \begin{bmatrix} 1 & \varrho \\ \varrho & 1 \end{bmatrix} \qquad \Longrightarrow \qquad \boldsymbol{C}_{UU} = \sigma_S^2 \begin{bmatrix} 1+\varrho & 0 \\ 0 & 1-\varrho \end{bmatrix}$$

Numerical Algorithms for Eigenvector Computation

Classical Jacobi Algorithm (Carl Gustav Jacob Jacobi, 1846)

- Iterative technique for (small) symmetric matrices
- Diagonalize matrix $\boldsymbol{C}$ by multiplication with elementary rotation matrices $\boldsymbol{R}$

$$\boldsymbol{C}^{(0)} = \boldsymbol{C} \tag{8}$$

$$\boldsymbol{C}^{(k+1)} = \boldsymbol{R}_k \, \boldsymbol{C}^{(k)} \, \boldsymbol{R}_k^{\mathrm{T}} = \underbrace{\boldsymbol{R}_k \, \boldsymbol{R}_{k-1} \cdots \boldsymbol{R}_0}_{\boldsymbol{A}_k} \, \boldsymbol{C} \, \underbrace{\boldsymbol{R}_0^{\mathrm{T}} \cdots \boldsymbol{R}_{k-1}^{\mathrm{T}} \cdots \boldsymbol{R}_k^{\mathrm{T}}}_{\boldsymbol{A}_k^{\mathrm{T}}} \tag{9}$$

with

$$\boldsymbol{R}_k = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & \cos\alpha & \cdots & \sin\alpha & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & & -\sin\alpha & \cdots & \cos\alpha & \cdots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \tag{10}$$

Numerical Algorithms for Eigenvector Computation

Classical Jacobi Algorithm

- Consider $\boldsymbol{R}_k$ with cos and sin terms in p-th and q-th row/column

➡ Transformation $\boldsymbol{R}_k\, \boldsymbol{C}^{(k)}\, \boldsymbol{R}_k^{\mathrm{T}}$ has following effect $(n, m \neq p, q)$

$$c_{pp}^{(k+1)} = c_{pp}^{(k)} \cos^2\alpha + 2c_{pq}^{(k)} \cos\alpha \sin\alpha + c_{qq}^{(k)} \sin^2\alpha \tag{11}$$

$$c_{qq}^{(k+1)} = c_{pp}^{(k)} \sin^2\alpha - 2c_{pq}^{(k)} \cos\alpha \sin\alpha + c_{qq}^{(k)} \cos^2\alpha \tag{12}$$

$$c_{pq}^{(k+1)} = c_{qp}^{(k+1)} = \left(c_{qq}^{(k)} - c_{pp}^{(k)}\right) \cos\alpha \sin\alpha + c_{pq}^{(k)} \left(\cos^2\alpha - \sin^2\alpha\right) \tag{13}$$

$$c_{pn}^{(k+1)} = c_{np}^{(k+1)} = c_{qn}^{(k)} \sin\alpha + c_{pn}^{(k)} \cos\alpha \tag{14}$$

$$c_{qn}^{(k+1)} = c_{nq}^{(k+1)} = c_{qn}^{(k)} \cos\alpha - c_{pn}^{(k)} \sin\alpha \tag{15}$$

$$c_{nm}^{(k+1)} = c_{mn}^{(k)} \tag{16}$$

➡ Determine element $c_{pq}^{(k)}$ with maximum absolute value and choose α in a way that the Jacobi rotation yields $c_{pq}^{(k+1)} = 0$

Numerical Algorithms for Eigenvector Computation

Classical Jacobi Algorithm

- Choosing α so that we get $c_{pq}^{(k+1)} = 0$

$$(c_{qq}^{(k)} - c_{pp}^{(k)}) \cos\alpha \sin\alpha + c_{pq}^{(k)} (\cos^2\alpha - \sin^2\alpha) = 0 \tag{17}$$

- Using trigonometric identities

$$2 \cos\alpha \sin\alpha = \sin(2\alpha) \qquad \text{and} \qquad \cos^2\alpha - \sin^2\alpha = \cos(2\alpha) \tag{18}$$

yields

$$\frac{1}{2} (c_{pp}^{(k)} - c_{qq}^{(k)}) \sin(2\alpha) \;=\; c_{pq}^{(k)} \cos(2\alpha)$$

$$\tan(2\alpha) \;=\; \frac{2\, c_{pq}^{(k)}}{c_{pp}^{(k)} - c_{qq}^{(k)}} \qquad \left(\alpha = \frac{\pi}{4} \quad \text{if} \quad c_{qq}^{(k)} = c_{pp}^{(k)}\right) \tag{19}$$

➡ Apply elementary Jacobi rotations until $\boldsymbol{C}^{(k)}$ becomes a diagonal matrix (in terms of required precision)

Numerical Algorithms for Eigenvector Computation

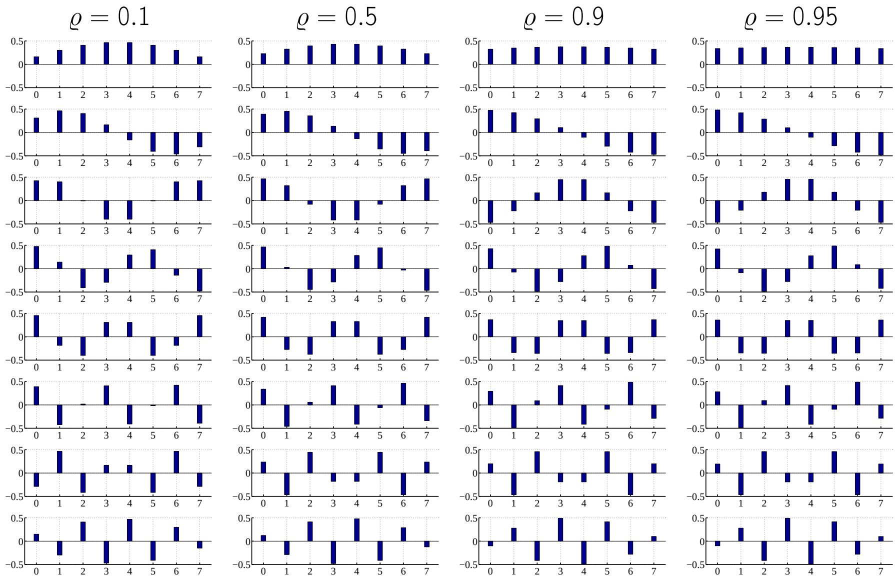
Classical Jacobi Algorithm

- Conceptually simple algorithm
- Slow convergence
- Unsuitable for large matrices

Numerous Advanced Numerical Algorithms

- Particularly fast algorithms for real symmetric matrices
- Typical: Two steps
  1. Transform matrix into Hessenberg / tridiagonal form
  2. Determine eigenvectors of simpler matrix using fast algorithms
- Some examples
  - Given rotations + divide-and-conquer
  - Householder transformation + QR algorithm
  - Householder transformation + MRRR algorithm

KLT Basis Functions for AR(1) Sources and $N = 8$



Optimal Orthogonal Transform for Gaussian Sources

Gaussian sources

- Each linear combination of Gaussian random vectors is another Gaussian random vector
- All transform coefficients have a Gaussian distribution
- All component quantizers can be described by the same distortion-rate function

$$D_k(R_k) = \sigma_k^2 \cdot g(R_k) \tag{20}$$

Transform Coding of Gaussian Sources

- Consider any orthogonal transform matrix $\boldsymbol{A}_k$

➞ Overall distortion-rate function

$$D(R, \boldsymbol{A}_k) = \sum_{k=0}^{N-1} \sigma_k^2(\boldsymbol{A}_k) \cdot g(R_k) \tag{21}$$

- Given quantizers: $D(R)$ depends on rate distribution and transform matrix

Optimal Orthogonal Transform for Gaussian Sources

- Consider an arbitrary orthogonal matrix $\boldsymbol{A}_0$ and an arbitrary rate distribution

$$(R_0, R_1, \cdots, R_{N-1}) \qquad \text{with} \qquad \sum_{k=0}^{N-1} R_k = R \tag{22}$$

Iterative algorithm with iterations $\boldsymbol{A}_k \mapsto \boldsymbol{A}_{k+1}$

1. Choose permutation matrix $\boldsymbol{P}_k$ that minimizes overall distortion

$$D(R, \boldsymbol{P}_k\boldsymbol{A}_k) = \sum_{k=0}^{N-1} \sigma_k^2(\boldsymbol{P}_k\boldsymbol{A}_k) \cdot g(R_k) \tag{23}$$

2. Apply Jacobi rotation

$$\boldsymbol{A}_{k+1} = \boldsymbol{R}_k \, \boldsymbol{P}_k \, \boldsymbol{A}_k \tag{24}$$

with $\boldsymbol{R}_k$ being the elementary rotation matrix that sets the maximum absolute value on a secondary diagonal of $\boldsymbol{C}_k$ equal to 0

$$\boldsymbol{C}_k = \boldsymbol{P}_k \, \boldsymbol{A}_k \, \boldsymbol{C}_{SS} \, \boldsymbol{A}_k^{\mathrm{T}} \, \boldsymbol{P}_k^{\mathrm{T}} \tag{25}$$

Optimal Orthogonal Transform for Gaussian Sources

Impact of Permutation on Distortion

- Obvious: Will never increase distortion

$$D(R, \boldsymbol{P}_k \boldsymbol{A}_k) \leq D(R, \boldsymbol{A}_k) \tag{26}$$

- Consider any pair of component rates with

$$g(R_i) \leq g(R_j) \tag{27}$$

- Overall distortion

$$\begin{aligned} D(R, \boldsymbol{P}_k \boldsymbol{A}_k) &= \cdots + \sigma_i^2 \cdot g(R_i) + \sigma_j^2 \cdot g(R_j) \\ &= \cdots + \sigma_i^2 \cdot g(R_i) + \sigma_j^2 \cdot g(R_i) + \sigma_j^2 \cdot (g(R_j) - g(R_i)) \\ &= \cdots + (\sigma_i^2 + \sigma_j^2) \cdot g(R_i) + \sigma_j^2 \cdot \underbrace{(g(R_j) - g(R_i))}_{\geq 0} \end{aligned} \tag{28}$$

- Hence, we have

$$g(R_i) \leq g(R_j) \quad \Longrightarrow \quad \sigma_i^2(\boldsymbol{P}_k \boldsymbol{A}_k) \geq \sigma_j^2(\boldsymbol{P}_k \boldsymbol{A}_k) \tag{29}$$

(otherwise distortion can be reduced by permuting matrix rows)

Optimal Orthogonal Transform for Gaussian Sources

Impact of Jacobi Rotation on Distortion

- Jacobi rotation does only change the variance for two coefficients (i, j)
- For $\sigma_i^2(\boldsymbol{P}_k \boldsymbol{A}_k) \geq \sigma_j^2(\boldsymbol{P}_k \boldsymbol{A}_k)$, we have

$$\sigma_i^2(\boldsymbol{A}_{k+1}) = \sigma_i^2(\boldsymbol{P}_k \boldsymbol{A}_k) + \delta(\boldsymbol{P}_k \boldsymbol{A}_k) \tag{30}$$

$$\sigma_j^2(\boldsymbol{A}_{k+1}) = \sigma_j^2(\boldsymbol{P}_k \boldsymbol{A}_k) - \delta(\boldsymbol{P}_k \boldsymbol{A}_k) \tag{31}$$

with

$$\delta(\boldsymbol{P}_k \boldsymbol{A}_k) = \frac{2c_{ij}^2}{(c_{ii} - c_{jj}) + \sqrt{(c_{ii} - c_{jj})^2 + 4\,c_{ij}^2}} \geq 0 \tag{32}$$

and c_{ij} being the elements of the covariance matrix

$$\boldsymbol{P}_k \cdot \boldsymbol{A}_k \cdot \boldsymbol{C}_{SS} \cdot \boldsymbol{A}_k^{\mathrm{T}} \cdot \boldsymbol{P}_k^{\mathrm{T}} \tag{33}$$

- Hence: Greater variance will never be decreased, small variance will never be increased

Optimal Orthogonal Transform for Gaussian Sources

Impact of One Iteration Step on Overall Distortion

- Permutation and Jacobi rotation ($\boldsymbol{A}_k \mapsto \boldsymbol{A}_{k+1}$) yield

$$\begin{aligned} D(R, \boldsymbol{A}_{k+1}) &= \sum_{n=0}^{N-1} \sigma_n^2(\boldsymbol{A}_{k+1}) \cdot g(R_n) \\ &= D(R, \boldsymbol{P}_k \boldsymbol{A}_k) + \delta(\boldsymbol{P}_k \boldsymbol{A}_k) \cdot \underbrace{(g(R_i) - g(R_j))}_{\leq 0} \\ &\leq D(R, \boldsymbol{P}_k \boldsymbol{A}_k) \leq D(R, \boldsymbol{A}_k) \end{aligned} \tag{34}$$

➡ Distortion will never be increased in any iteration step

- Algorithm represents classical Jacobi algorithm with additional permutations (no effect on resulting basis vectors, but only their order)

➡ Algorithm approaches KLT transform matrix

➡ **For Gaussian sources and MSE distortion, KLT is the optimal orthogonal transform**

Asymp. High-Rate Performance of KLT for Gaussian Sources

- Transform coefficient variances σ_k^2 are equal to the eigenvalues ξ_i of $\boldsymbol{C}_{SS}$
- High-rate approximation for Gaussian source and optimal ECSQ

$$\begin{aligned} D_{\mathrm{KLT}}(R) &= \frac{\pi e}{6} \cdot \tilde{\sigma}^2 \cdot 2^{-2R} = \frac{\pi e}{6} \cdot \tilde{\xi} \cdot 2^{-2R} \\ &= \frac{\pi e}{6} \cdot 2^{\frac{1}{N}\sum_{k=0}^{N-1} \log_2 \xi_k} \cdot 2^{-2R} \end{aligned} \tag{35}$$

- For $N \to \infty$, we can apply the theorem of Szegö and Grenander

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} G(\xi_k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(\Phi(\omega))\, \mathrm{d}\omega \tag{36}$$

➡ Resulting distortion-rate function for KLT of infinite size for high rates

$$D_{\mathrm{KLT}}^{\infty}(R) = \frac{\pi e}{6} \cdot 2^{\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \log_2 \Phi_{SS}(\omega)\, \mathrm{d}\omega\right)} \cdot 2^{-2R} \tag{37}$$

Asymp. High-Rate Performance of KLT for Gaussian Sources

- Asymptotic distortion-rate function for KLT of infinite size for high rates

$$D^{\infty}_{\mathrm{KLT}}(R) = \varepsilon^2 \cdot 2^{\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \log_2 \Phi_{SS}(\omega)\, \mathrm{d}\omega \right)} \cdot 2^{-2R} \qquad (38)$$

- Information distortion-rate function (fundamental bound) is by a factor $\varepsilon^2 = \pi e/6$ smaller

$$D(R) = 2^{\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \log_2 \Phi_{SS}(\omega)\, \mathrm{d}\omega \right)} \cdot 2^{-2R} \qquad (39)$$

Transform Gain

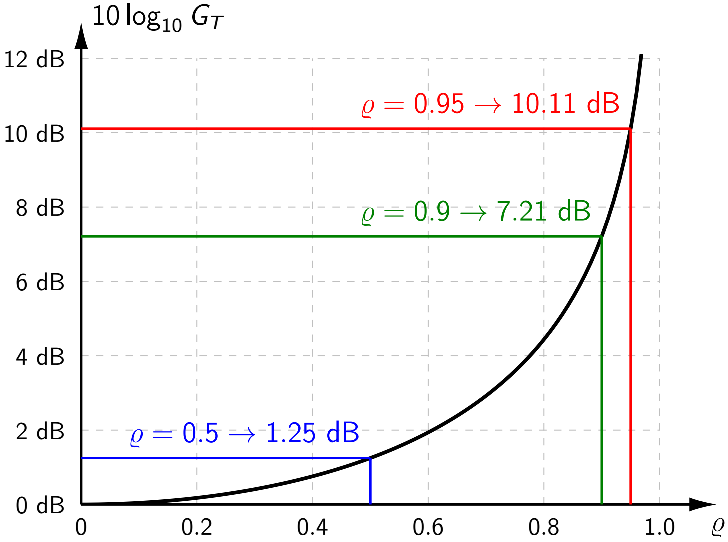
- Asymptotic transform gain ($N \to \infty$) at high rates

$$G^{\infty}_{\mathrm{T}} = \frac{D_{\mathrm{SC}}(R)}{D_{\mathrm{TC}}(R)} = \frac{\varepsilon^2\, \sigma_S^2\, 2^{-2R}}{D^{\infty}_{\mathrm{KLT}}(R)} = \frac{\frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_{SS}(\omega) d\omega}{2^{\frac{1}{2\pi} \int_{-\pi}^{\pi} \log_2 \Phi_{SS}(\omega) d\omega}} \qquad (40)$$

- Asymptotic transform gain for Gauss-Markov sources

$$G^{\infty}_{\mathrm{T}} = \frac{1}{1 - \varrho^2} \qquad (41)$$

Asymptotic Transform Gain for Gauss-Markov Processes



KLT for Gauss-Markov at High Rates

- High rate distortion-rate function for KLT of size N

$$D(R) = \varepsilon^2 \cdot \tilde{\sigma} \cdot 2^{-2R} = \varepsilon^2 \cdot \tilde{\xi} \cdot 2^{-2R} \tag{42}$$

- Remember: Product of eigenvalues = determinate

$$\tilde{\xi} = \left(\prod_{k=0}^{N-1} \xi_k \right)^{\frac{1}{N}} = |\boldsymbol{C}_N|^{\frac{1}{N}} \tag{43}$$

- Determinant of Gauss-Markov source

$$|\boldsymbol{C}_N| = \begin{vmatrix} \sigma_S^2 & \varrho \cdot \sigma_S^2 & \varrho^2 \cdot \sigma_S^2 & \cdots & \varrho^{N-1} \cdot \sigma_S^2 \\ \varrho \cdot \sigma_S^2 & \sigma_S^2 & \varrho \cdot \sigma_S^2 & \cdots & \varrho^{N-2} \cdot \sigma_S^2 \\ \varrho^2 \cdot \sigma_S^2 & \varrho \cdot \sigma_S^2 & \sigma_S^2 & \cdots & \varrho^{N-3} \cdot \sigma_S^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varrho^{N-1} \cdot \sigma_S^2 & \varrho^{N-2} \cdot \sigma_S^2 & \varrho^{N-3} \cdot \sigma_S^2 & \cdots & \sigma_S^2 \end{vmatrix} \tag{44}$$

KLT for Gauss-Markov at High Rates

- Expand determinant along the first columns using Laplace's formula

$$|\boldsymbol{C}_N| = \sum_{k=0}^{N-1} (-1)^k \, \phi_{k,0} \, |\boldsymbol{C}_N^{(k,0)}| = \sum_{k=0}^{N-1} (-1)^k \, \sigma_S^2 \, \varrho^k \, |\boldsymbol{C}_N^{(k,0)}| \tag{45}$$

with $\boldsymbol{C}_N^{(k,\ell)}$ being the matrix that is obtained by removing the k-th row and ℓ-th column from $\boldsymbol{C}_N$

- Consider matrices $\boldsymbol{C}_N^{(k,0)}$ with $k > 1$
  - First row is equal to second row multiplied by ϱ
  - First row is linearly dependent of second row and, hence, we have

$$\forall k > 1, \qquad |\boldsymbol{C}_N^{(k,0)}| = 0 \tag{46}$$

➡ Above formula simplifies to

$$|\boldsymbol{C}_N| = \sigma_S^2 \, |\boldsymbol{C}_N^{(0,0)}| - \sigma_S^2 \, \varrho \, |\boldsymbol{C}_N^{(1,0)}| \tag{47}$$

KLT for Gauss-Markov at High Rates

- Matrix $\boldsymbol{C}_N^{(0,0)}$ is equal to $\boldsymbol{C}_{N-1}$
- Matrix $\boldsymbol{C}_N^{(1,0)}$ has the form

$$\boldsymbol{C}_N^{(1,0)} = \left| \begin{array}{ccccc} \varrho \cdot \sigma_S^2 & \varrho^2 \cdot \sigma_S^2 & \varrho^3 \cdot \sigma_S^2 & \cdots & \varrho^{N-1} \cdot \sigma_S^2 \\ \varrho \cdot \sigma_S^2 & \sigma_S^2 & \varrho \cdot \sigma_S^2 & \cdots & \varrho^{N-3} \cdot \sigma_S^2 \\ \varrho^2 \cdot \sigma_S^2 & \varrho \cdot \sigma_S^2 & \sigma_S^2 & \cdots & \varrho^{N-4} \cdot \sigma_S^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varrho^{N-2} \cdot \sigma_S^2 & \varrho^{N-3} \cdot \sigma_S^2 & \varrho^{N-4} \cdot \sigma_S^2 & \cdots & \sigma_S^2 \end{array} \right| \tag{48}$$

➡ Same as $\boldsymbol{C}_{N-1}$ except that first row is multiplied by ϱ

➡ Determinate is given by

$$|\boldsymbol{C}_N^{(1,0)}| = \varrho\, |\boldsymbol{C}_{N-1}| \tag{49}$$

➡ Recursive formula

$$\begin{aligned} |\boldsymbol{C}_N| &= \sigma_S^2\, |\boldsymbol{C}_N^{(0,0)}| - \sigma_S^2\, \varrho\, |\boldsymbol{C}_N^{(1,0)}| \\ &= \sigma_S^2\, (1 - \varrho^2)\, |\boldsymbol{C}_{N-1}| \end{aligned} \tag{50}$$

KLT for Gauss-Markov at High Rates

- Recursive formula for determinate of autocovariance matrix

$$|\boldsymbol{C}_N| = \sigma_S^2 \, (1 - \varrho^2) \, |\boldsymbol{C}_{N-1}| \tag{51}$$

- Since $|\boldsymbol{C}_1| = \sigma_S^2$, we obtain

$$|\boldsymbol{C}_N| = \sigma_S^{2N} \, (1 - \varrho^2)^{N-1} \tag{52}$$

and

$$\tilde{\sigma}^2 = |\boldsymbol{C}_N|^{\frac{1}{N}} = \sigma_S^2 \, (1 - \varrho^2)^{\frac{N-1}{N}} \tag{53}$$

➡ High rate distortion-rate function for KLT + ECSQ

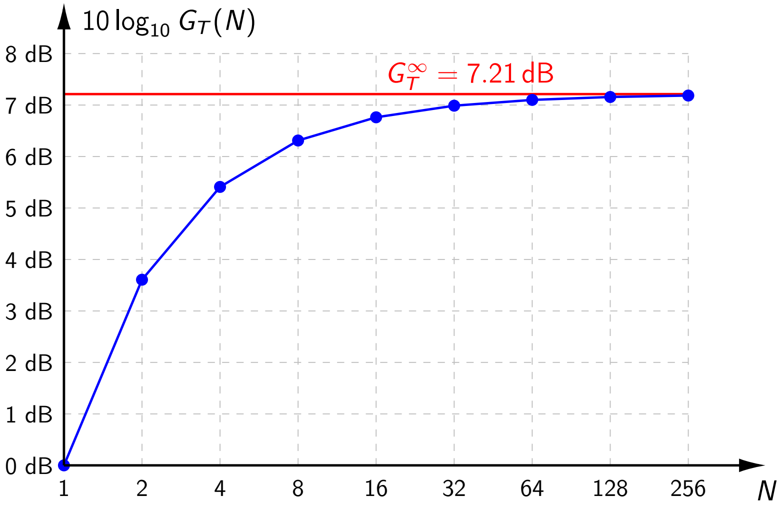
$$D_N(R) = \frac{\pi e}{6} \, \sigma_S^2 \, (1 - \varrho^2)^{\frac{N-1}{N}} \, 2^{-2R} \tag{54}$$

➡ High rate transform coding gain for KLT + ECSQ

$$G_T^N = (1 - \varrho^2)^{\frac{1-N}{N}} \tag{55}$$

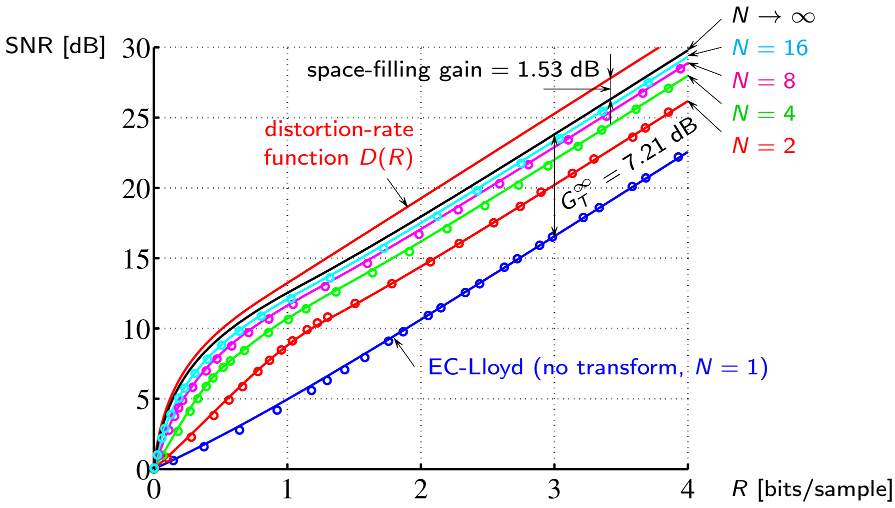
KLT for Gauss-Markov at High Rates

- High rate transform coding gain for Gauss-Markov with $\varrho = 0.9$



Experiment: KLT Coding of Gauss-Markov ($\varrho = 0.9$)

➡ Better performance than DPCM at low rates



Summary

Optimal Unitary/Orthogonal Transform

- Difficult to determine due to interdependencies of transform and quantization
- In general, optimal transform depends on bit rate
- Can be determined using iterative algorithm

Karhunen Loève Transform (KLT)

- Orthogonal transform that yields decorrelated transform coefficients
- Transform matrix: Matrix of unit-norm eigenvectors
- KLT minimizes the geometric mean of transform coefficient variances
- KLT is signal dependent, but not bit rate dependent
- KLT is the optimal transform for Gaussian sources

High-Rate Transform Gain of KLT for Gaussian Sources

- Asymptotic ($N \to \infty$): 1.53 dB worse than fundamental lower bound
- Gauss-Markov: Transform gain increases with block size N