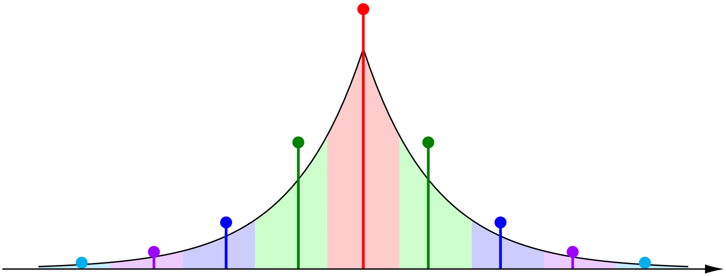
Quantization II



Entropy-Constrained Scalar Quantization (ECSQ)

- Lloyd quantizer: Minimize distortion for given number K of intervals
- Now: Consider quantizer design with variable-length coding of indices

➡ Average rate (without exploiting dependencies between quantization indices)

$$R = \sum_{\forall k} p_k \cdot \ell_k \;\geq\; H(S') = -\sum_{\forall k} p_k \log_2 p_k \tag{1}$$

with

$$p_k = \int_{u_k}^{u_{k+1}} f(s)\,\mathrm{d}s \tag{2}$$

➡ Consider entropy instead of the rate of an actual code

➡ Average MSE distortion

$$D = \mathrm{E}\{\, d(S, Q(S))\,\} = \sum_{\forall k} \int_{u_k}^{u_{k+1}} (s - s'_k)^2 \cdot f(s)\,\mathrm{d}s \tag{3}$$

Joint Minimization of Distortion and Rate/Entropy

- We look for solutions of constrained minimization problems

$$\min D \qquad \text{subject to} \qquad R \leq R_{\text{target}} \tag{4}$$

or equivalently

$$\min R \qquad \text{subject to} \qquad D \leq D_{\text{target}} \tag{5}$$

➡ Convert constrained optimization problem into unconstrained problem

➡ Minimized Lagrange function J (with Lagrange multiplier $\lambda > 0$)

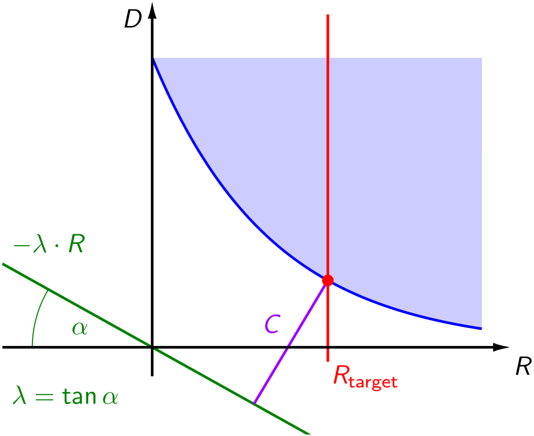
$$J = D + \lambda \cdot R \tag{6}$$

$$= \mathrm{E}\{\, d(S, S') \,\} + \lambda \cdot \mathrm{E}\{\, \ell(S') \,\} \tag{7}$$

$$= \mathrm{E}\{\, d(S, S') \,\} + \lambda \cdot H(S') \tag{8}$$

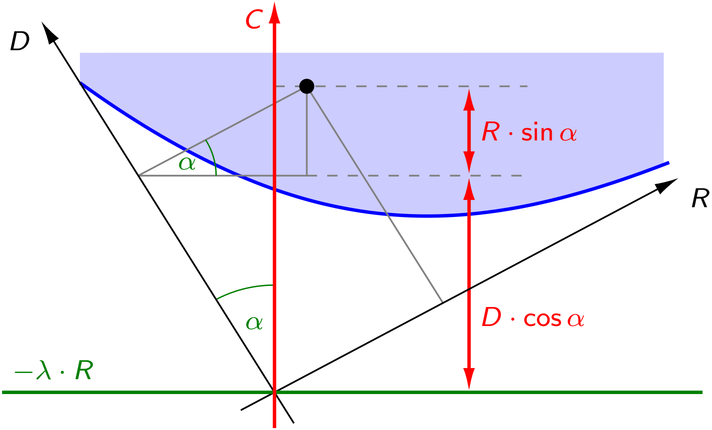
- Each Lagrange multiplier $\lambda > 0$ corresponds to a rate constraint R_{target} (or distortion constraint D_{target})

Lagrangian Optimization



- ➡ Points on convex hull: Minimize distance C to line $D = -\lambda \cdot R$
- ➡ Geometrical interpretation: Rotate coordinate system by angle α

Lagrangian Optimization



- ➡ Minimize distance: $C = D \cdot \cos\alpha + R \cdot \sin\alpha$
- ➡ Equivalent minimization: $J = D + \lambda \cdot R$ (note: $\lambda = \tan\alpha$)

Optimal Scalar Quantizer

- Minimize Lagrange function

$$
\begin{aligned}
J &= D + \lambda \cdot R \\
&= \mathrm{E}\{\, d(S, S') \,\} + \lambda \cdot \mathrm{E}\{\, \ell(S') \,\} \\
&= \mathrm{E}\{\, d(S,\, \beta(\alpha(S))\,) \,\} + \lambda \cdot \mathrm{E}\{\, \ell(\, \alpha(S)\,) \,\} \\
&= \sum_{\forall k} \int_{u_k}^{u_{k+1}} d(s, s'_k)\, f(s)\, \mathrm{d}s + \lambda \cdot \sum_{\forall k} \ell_k \int_{u_k}^{u_{k+1}} f(s)\, \mathrm{d}s
\end{aligned}
\tag{9}
$$

➡ Lagrange parameter λ determines trade-off between rate and distortion

➡ Optimal design: Minimize $J = D + \lambda R$ with respect to

- ➡ Reconstruction levels s'_k (decoder mapping β)
- ➡ Decision thresholds u_k (encoding mapping α)
- ➡ Codeword lengths ℓ_k (lossless mapping γ)

Optimal Decoder Mapping: Centroid Condition

$$J = \sum_{\forall k} \int_{u_k}^{u_{k+1}} d(s, s'_k)\, f(s)\, \mathrm{d}s \;+\; \lambda \cdot \sum_{\forall k} \ell_k \int_{u_k}^{u_{k+1}} f(s)\, \mathrm{d}s$$

1 Optimize reconstruction levels s'_k
for given decision thresholds u_k (and codeword lengths ℓ_k)

- Note: Rate term does not depend on reconstruction levels s'_k

➡ **General centroid condition** (same as for Lloyd quantizer)

$$\boxed{s'_k = \arg\min_{s'} \mathrm{E}\Big\{ d(S, s') \mid S \in \mathcal{C}_k \Big\}} \qquad (10)$$

➡ **Centroid condition for MSE distortion**

$$\boxed{s'_k = \mathrm{E}\{ S \mid S \in \mathcal{C}_k \} = \frac{1}{p_k} \int_{u_k}^{u_{k+1}} s\, f(s)\, \mathrm{d}s = \frac{\int_{u_k}^{u_{k+1}} s\, f(s)\, \mathrm{d}s}{\int_{u_k}^{u_{k+1}} f(s)\, \mathrm{d}s}} \qquad (11)$$

Optimal Lossless Mapping: Entropy Condition

$$J = \sum_{\forall k} \int_{u_k}^{u_{k+1}} d(s, s'_k)\, f(s)\, \mathrm{d}s \; + \; \lambda \cdot \sum_{\forall k} \ell_k \int_{u_k}^{u_{k+1}} f(s)\, \mathrm{d}s$$

2 Optimize codeword lengths ℓ_k
for given decision thresholds u_k (and reconstruction levels s'_k)

- Note: Distortion term does not depend on codeword lengths ℓ_k

➡ Remember lossless coding: $R \geq H(S')$

$$\sum_{\forall k} p_k \cdot \ell_k \geq -\sum_{\forall k} p_k \cdot \log_2 p_k \qquad \text{(equality iff } \ell_k = -\log_2 p_k\text{)} \tag{12}$$

➡ **Entropy condition** (neglecting inefficiency of actual entropy coding)

$$\boxed{\ell_k = -\log_2 p_k = -\log_2 \left(\int_{u_k}^{u_{k+1}} f(s)\, \mathrm{d}s \right)} \tag{13}$$

Optimal Encoder Mapping

$$J = \sum_{\forall k} \int_{u_k}^{u_{k+1}} d(s, s'_k)\, f(s)\, \mathrm{d}s \;+\; \lambda \cdot \sum_{\forall k} \ell_k \int_{u_k}^{u_{k+1}} f(s)\, \mathrm{d}s$$

3 Optimize decision thresholds u_k
for given reconstruction levels s'_k and codeword lengths ℓ_k

➡ Map each input value s to interval $\mathcal{C}_k$ that minimizes $d(s, s'_k) + \lambda \cdot \ell_k$

$$\alpha(s) = \arg\min_{\forall s'_k} \; d(s, s'_k) + \lambda \cdot \ell_k \tag{14}$$

- Note: Each threshold u_k impacts only neighboring intervals $\mathcal{C}_{k-1}$ and $\mathcal{C}_k$

➡ **Modified nearest neighbor condition**

$$\boxed{d(u_k, s'_{k-1}) + \lambda \cdot \ell_{k-1} = d(u_k, s'_k) + \lambda \cdot \ell_k} \tag{15}$$

Optimal Encoder Mapping: MSE Distortion

- Optimal encoding mapping for MSE distortion

$$\begin{aligned}
d(u_k, s'_{k-1}) + \lambda \cdot \ell_{k-1} &= d(u_k, s'_k) + \lambda \cdot \ell_k \\
(u_k - s'_{k-1})^2 + \lambda \cdot \ell_{k-1} &= (u_k - s'_k)^2 + \lambda \cdot \ell_k \\
u_k^2 - 2u_k s'_{k-1} + (s'_{k-1})^2 + \lambda \cdot \ell_{k-1} &= u_k^2 - 2u_k s'_k + (s'_k)^2 + \lambda \cdot \ell_k \\
2u_k(s'_k - s'_{k-1}) &= (s'_k)^2 - (s'_{k-1})^2 + \lambda\,(\ell_k - \ell_{k-1}) \\
2u_k(s'_k - s'_{k-1}) &= (s'_k - s'_{k-1})(s'_k + s'_{k-1}) + \lambda\,(\ell_k - \ell_{k-1})
\end{aligned}$$

➡ **Optimal decision thresholds for MSE distortion**

$$\boxed{u_k = \frac{1}{2}\left(s'_{k-1} + s'_k\right) + \frac{\lambda}{2}\left(\frac{\ell_k - \ell_{k-1}}{s'_k - s'_{k-1}}\right)} \qquad (16)$$

➡ **Threshold is shifted towards the reconstruction level with the longer codeword (less probable interval)**

Optimal Entropy-Constrained Scalar Quantizer (ECSQ)

Necessary conditions for optimality

- Centroid condition (for reconstruction levels s'_k)

$$s'_k = \arg\min_{s' \in \mathbb{R}} \mathrm{E}\Big\{ d(S, s') \mid S \in \mathcal{C}_k \Big\} \tag{17}$$

- Entropy condition (for codeword lengths ℓ_k)

$$\ell_k = -\log_2 p_k = -\log_2 \int_{u_k}^{u_{k+1}} f(s)\, \mathrm{d}s \tag{18}$$

- Modified nearest neighbor condition (for decision threshold u_k)

$$d(u_k, s'_{k-1}) + \lambda \cdot \ell_{k-1} = d(u_k, s'_k) + \lambda \cdot \ell_k \tag{19}$$

Optimal Entropy-Constrained Scalar Quantizer (ECSQ)

Optimality conditions for MSE distortion

- Centroid condition (for reconstruction levels s'_k)

$$s'_k = \frac{\int_{u_k}^{u_{k+1}} s\, f(s)\, \mathrm{d}s}{\int_{u_k}^{u_{k+1}} f(s)\, \mathrm{d}s} \tag{20}$$

- Entropy condition (for codeword lengths ℓ_k)

$$\ell_k = -\log_2 p_k \tag{21}$$

- Modified nearest neighbor condition (for decision threshold u_k)

$$u_k = \frac{1}{2}\left(s'_{k-1} + s'_k\right) + \frac{\lambda}{2}\left(\frac{\ell_k - \ell_{k-1}}{s'_k - s'_{k-1}}\right) \tag{22}$$

Design of optimal entropy-constrained scalar quantizers

- In general: Cannot be derived in closed form
- Iterative algorithm similar to Lloyd algorithm

Entropy-Constrained Lloyd Algorithm for Given Pdf (MSE)

Given is:
- the marginal probability density function $f(s)$ of the source
- a Lagrange multiplier $\lambda > 0$

Iterative quantizer design

1. Choose an initial set of reconstruction levels $\{s'_k\}$ and codeword lengths $\{\ell_k\}$
2. Update the decision thresholds $\{u_k\}$ according to

$$u_k = \frac{1}{2}\left(s'_{k-1} + s'_k\right) + \frac{\lambda}{2}\left(\frac{\ell_k - \ell_{k-1}}{s'_k - s'_{k-1}}\right)$$

3. Update the reconstruction levels $\{s'_k\}$ according to

$$s'_k = \frac{\int_{u_k}^{u_{k+1}} s\, f(s)\, \mathrm{d}s}{\int_{u_k}^{u_{k+1}} f(s)\, \mathrm{d}s}$$

4. Update the codeword lengths $\{\ell_k\}$ according to

$$\ell_k = -\log_2 p_k$$

5. Repeat the previous three steps until convergence

Entropy-Constrained Lloyd Algorithm for a Training Set

Given is:
- a sufficiently large realization $\{s_n\}$ of the considered source
- a Lagrange multiplier $\lambda > 0$

Iterative quantizer design

1. Choose an initial set of reconstruction levels $\{s'_k\}$ and codeword lengths $\{\ell_k\}$
2. Associate all samples of the training set $\{s_n\}$ with one of the quantization intervals $\mathcal{C}_k$ according to

$$\alpha(s_n) = \arg\min_{\forall k} \; d(s_n, s'_k) + \lambda \cdot \ell_k$$

3. Update the reconstruction levels $\{s'_k\}$ according to

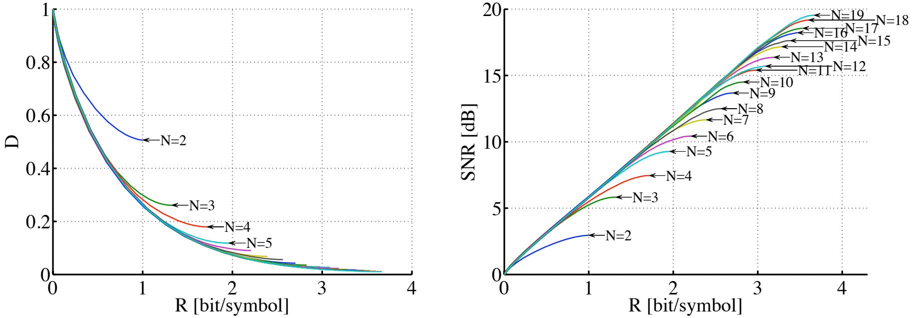
$$s'_k = \arg\min_{s' \in \mathbb{R}} \; \mathrm{E}\{\, d(S, s') \mid \alpha(S) = k \,\}$$

4. Update the codeword lengths $\{\ell_k\}$ according to

$$\ell_k = -\log_2 p_k$$

5. Repeat the previous three steps until convergence

Number of Initial Intervals for EC Lloyd Algorithm

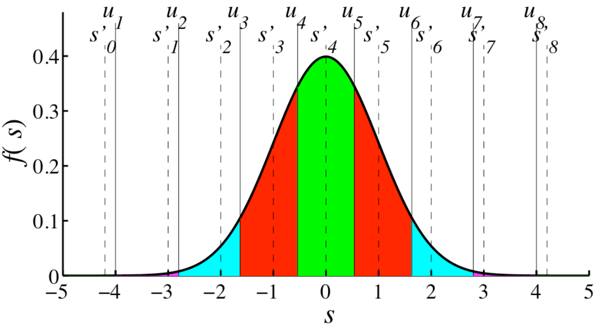


- Thresholds u_k are shifted towards less probable reconstruction levels
- Probability of less often chosen reconstruction levels is further reduced
- As a consequence, symbols get "removed" and the EC Lloyd algorithm can be initialized with more symbols than the final result

➡ **Use large number of intervals in initialization**

Entropy-Constrained Lloyd Algorithm for Gaussian Source

- Gaussian source with zero mean and unit variance
- Design optimal entropy-constrained quantizer with rate $R = 2$ bit/symbol

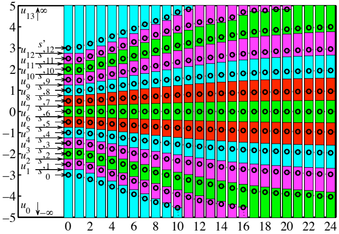


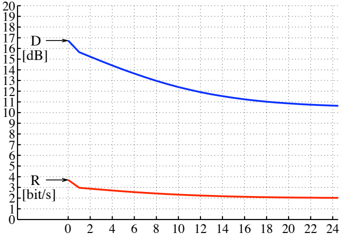
Comparison of MSE distortion (for $R = 2$ bits/sample)

- EC-Lloyd: $D = 0.09$ (SNR = 10.5 dB)
- Lloyd: $D = 0.12$ (SNR = 9.3 dB)
- SLB: $D = 0.0625$ (SNR = 12.0 dB)

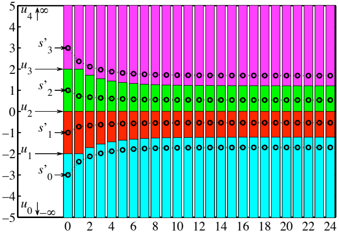
Convergence of EC Lloyd Algorithm for Gaussian Source

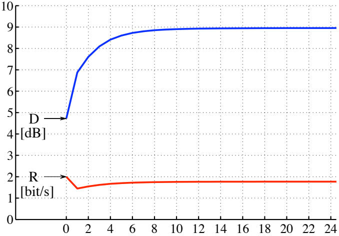
Initialization A:
$s'_k = -3 + 0.5 \cdot k$





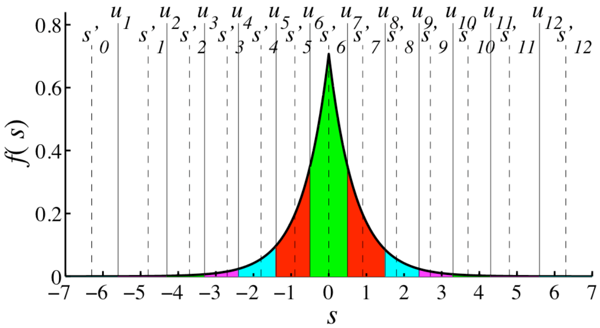
Initialization B:
$s'_k = -3 + 2 \cdot k$





Entropy-Constrained Lloyd Algorithm for Laplacian Source

- Laplacian source with zero mean and unit variance
- Design optimal entropy-constrained quantizer with rate $R = 2$ bit/symbol



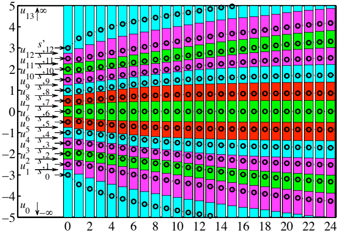
Comparison of MSE distortion (for $R = 2$ bits/sample)

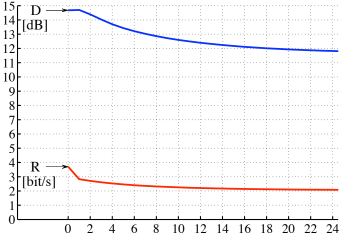
- EC-Lloyd: $D = 0.07$ (SNR = 11.5 dB)
- Lloyd: $D = 0.18$ (SNR = 7.6 dB)
- SLB: $D = 0.054$ (SNR = 12.7 dB)

Convergence of EC Lloyd Algorithm for Laplacian Source

Initialization A:

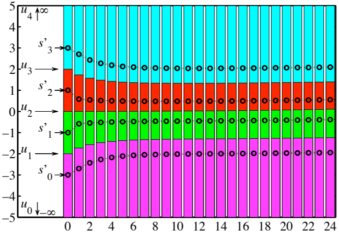
$$s'_k = -3 + 0.5 \cdot k$$

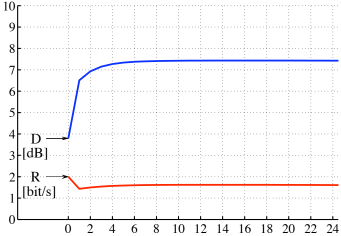




Initialization B:

$$s'_k = -3 + 2 \cdot k$$





Remember: High-Rate Approximations

Approximations for High Rates / Small Distortions

- Marginal pdf $f(s)$ is (nearly) constant inside each quantization interval

$$f(s) \approx f(s'_k) \qquad \forall s \in [u_k, u_{k+1}) \tag{23}$$

➡ Probability $p_k = \mathrm{P}(S \in \mathcal{C}_k)$ is given by

$$p_k = \int_{u_k}^{u_{k+1}} f(s)\,\mathrm{d}s = \Delta_k \cdot f(s'_k) \tag{24}$$

➡ Average distortion

$$D = \sum_{\forall k} \int_{u_k}^{u_{k+1}} d(s, s'_k)\, f(s)\,\mathrm{d}s = \sum_{\forall k} f(s'_k) \int_{u_k}^{u_{k+1}} d(s, s'_k)\,\mathrm{d}s \tag{25}$$

➡ Average MSE distortion for centroid quantizers

$$D = \frac{1}{12} \sum_{\forall k} p_k\, \Delta_k^2 \tag{26}$$

High-Rate Approximation: Optimal Quantizers with VLC

Average rate for optimal variable length coding (at high rates)

- Lossless source coding

$$R = H(S') = -\sum_{\forall k} p_k \, \log_2 p_k \qquad (27)$$

- High rate approximation $p_k = f(s'_k)\,\Delta_k$

$$\begin{aligned} R &= -\sum_{\forall k} p_k \left(\log_2 f(s'_k) + \log_2 \Delta_k \right) \\ &= -\sum_{\forall k} f(s'_k) \, \log_2 f(s'_k)\,\Delta_k - \frac{1}{2} \sum_{\forall k} p_k \, \log_2 \Delta_k^2 \end{aligned} \qquad (28)$$

- Asymptotic limit ($\Delta_k \to 0$) for first term

$$\begin{aligned} R &= -\int_{-\infty}^{\infty} f(s) \, \log_2 f(s)\,\mathrm{d}s - \frac{1}{2} \sum_{\forall k} p_k \, \log_2 \Delta_k^2 \\ &= h(S) - \frac{1}{2} \sum_{\forall k} p_k \, \log_2 \Delta_k^2 \end{aligned} \qquad (29)$$

High-Rate Approximation: Optimal Quantizers with VLC

- Will use: **Jensen's inequality for convex functions $\psi(x)$**

$$\sum_{\forall k} \alpha_k \, \psi(x_k) \geq \psi\left(\sum_{\forall k} \alpha_k \, x_k \right) \qquad \text{for} \qquad \sum_{\forall k} \alpha_k = 1 \tag{30}$$

with equality iff $x_k = \text{const}$

Average rate for optimal variable length coding (at high rates)

- Apply Jensen's inequality: Convex function $\psi(x) = -\log_2(x)$

$$\begin{aligned} R &= h(S) - \frac{1}{2} \sum_{\forall k} p_k \, \log_2 \Delta_k^2 \\ &\geq h(S) - \frac{1}{2} \log_2 \left(\sum_{\forall k} p_k \, \Delta_k^2 \right) \end{aligned} \tag{31}$$

with equality iff $\Delta_k = \text{const}$

High-Rate Approximation: Optimal Quantizers with VLC

Optimal scalar quantizers with variable length coding (at high rates)

- Remember: High-rate distortion approximation for centroid quantizers

$$D = \frac{1}{12} \sum_{\forall k} p_k \, \Delta_k^2$$

➡ **High-rate operational rate distortion function**

$$R_V(D) = h(S) - \frac{1}{2} \log_2 \left(\sum_{\forall k} p_k \, \Delta_k^2 \right) = h(S) - \frac{1}{2} \log_2(12\, D) \qquad (32)$$

➡ **High-rate operational distortion rate function** (Gish & Pierce, 1961)

$$D_V(R) = \frac{1}{12} \, 2^{2h(S)} \, 2^{-2R} \qquad (33)$$

➡ **For MSE distortion and high rates: Optimal scalar quantizer with variable length coding have uniform quantization step sizes**

Comparison to Shannon Lower Bound

- High-rate approximation for optimal quantizer with variable-length codes

$$D_V(R) = \varepsilon_V^2 \cdot \sigma^2 \cdot 2^{-2R} \qquad \text{with} \qquad \varepsilon_V^2 = \frac{1}{12\,\sigma^2} \cdot 2^{2h(S)}$$

- Shannon lower bound: High-rate approximation of rate-distortion function

$$D_L(R) = \varepsilon_L^2 \cdot \sigma^2 \cdot 2^{-2R} \qquad \text{with} \qquad \varepsilon_L^2 = \frac{1}{2\pi e\,\sigma^2} \cdot 2^{2h(S)}$$

➡ Distortion increase (at same rate R) relative to Shannon lower bound

$$\frac{D_V(R)}{D_L(R)} = \frac{\varepsilon_V^2}{\varepsilon_L^2} = \frac{\pi e}{6} \approx 1.42 \qquad \text{(Loss in SNR: 1.53 dB)}$$

➡ Rate increase (at same distortion D) relative to Shannon lower bound

$$\begin{aligned} R_V(D) - R_L(D) &= \left(h(S) - \frac{1}{2}\log_2\left(12D\right) \right) - \left(h(S) - \frac{1}{2}\log_2\left(2\pi e D\right) \right) \\ &= \frac{1}{2}\log_2 \frac{\pi e}{6} \approx 0.2546 \qquad (\approx 1/4 \text{ bit per sample}) \end{aligned}$$

Comparison of High-Rate Approximations

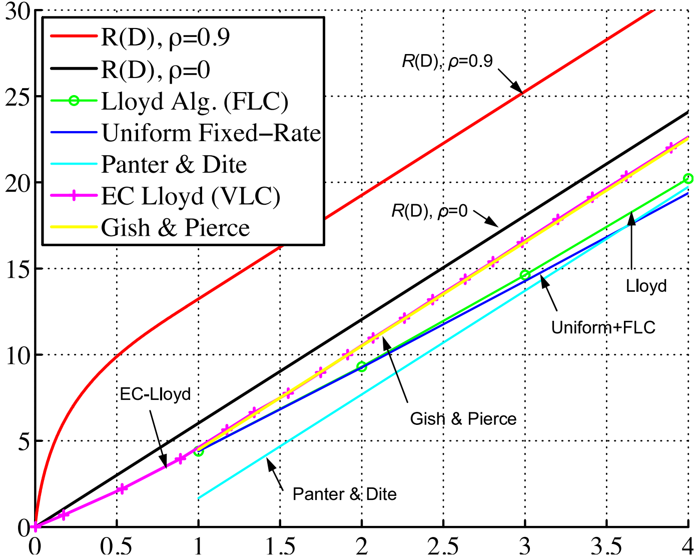
- Operational Distortion-rate function at high rates is given as

$$D_X(R) = \varepsilon_X^2 \cdot \sigma^2 \cdot 2^{-2R}$$

- Values of ε_X^2 depend on quantization method and pdf

Method	*Shannon Lower Bound (SLB)*	*Panter & Dite* (Lloyd Quant. & FLC)	*Gish & Pierce* (ECSQ & VLC)
Uniform pdf	$\frac{6}{\pi e} \approx 0.7$	1 (1.53 dB to SLB)	1 (1.53 dB to SLB)
Laplacian pdf	$\frac{e}{\pi} \approx 0.86$	$\frac{9}{2} = 4.5$ (7.1 dB to SLB)	$\frac{e^2}{6} \approx 1.23$ (1.53 dB to SLB)
Gaussian pdf	1	$\frac{\sqrt{3}\pi}{2} \approx 2.72$ (4.34 dB to SLB)	$\frac{\pi e}{6} \approx 1.42$ (1.53 dB to SLB)

Scalar Quantizers for Gaussian-Markov Sources



Summary

Lagrangian Optimization

- Re-formulate constrained optimization problems

$$\min_{\boldsymbol{p}} D(\boldsymbol{p}) \quad \text{subject to} \quad R(\boldsymbol{p}) \leq R_{\text{target}}$$

  as unconstrained optimization problem (with $\lambda \geq 0$)

$$\min_{\boldsymbol{p}} D(\boldsymbol{p}) + \lambda \cdot R(\boldsymbol{p})$$

- Each solution $\boldsymbol{p}_\lambda$ of the unconstrained problem does also represent a solution of the constrained problem with $R_{\text{target}} = R(\boldsymbol{p}_\lambda)$
- Lagrange multiplier λ determines trade-off between distortion and rate
- Geometrical interpretation:
  - Rotation of coordinate system by $\alpha = \arctan(\lambda)$
  - Conventional minimization in rotated space

Summary

Entropy-Constrained Scalar Quantizer (ECSQ)

- Optimal scalar quantizer (assuming optimal entropy coding)
- Three optimization criterions
  - Centroid condition (MSE): $s'_k = \left(\int_{u_k}^{u_{k+1}} s\, f(s)\, \mathrm{d}s \right) / \left(\int_{u_k}^{u_{k+1}} f(s)\, \mathrm{d}s \right)$
  - Entropy condition: $\ell_k = -\log_2 \left(\int_{u_k}^{u_{k+1}} f(s)\, \mathrm{d}s \right)$
  - Mod. neighbor cond. (MSE): $u_k = \frac{1}{2}\left(s'_k + s'_{k-1} \right) + \frac{\lambda}{2}\left(\frac{\ell_k - \ell_{k-1}}{s'_k - s'_{k-1}} \right)$
- Quantizer design: Iterative entropy-constrained Lloyd algorithm
  - Update decision thresholds u_k (for given s'_k and ℓ_k)
  - Update reconstruction levels s'_k and codeword lengths ℓ_k (for given u_k)
- May need large number ($K \to \infty$) of intervals for obtaining optimal quantizer
- Coding efficiency at high rates / small distortions
  - Distortion is factor 1.42 higher (1.53 dB) than SLB
  - Rate is roughly 0.25 bits per sample larger than SLB